Fluctuation Intensity of Passive Species in a Turbulent Subsonic Jet

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The equation for the turbulent fluctuation intensity of a passive scalar for incompressible shear flow was applied to a subsonic circular jet, using phenomenological diffusivities, for both the diffusion of the scalar mean and mean square fluctuation of the scalar, to determine the values of these diffusivities which give the best agreement between the calculated self-similar radial profile of the fluctuation intensity and experimental measurements. From experimental values in smoke jets and heated jets of the ratio of the fluctuation intensity to mean on the centerline and their radial profiles, it appears that the turbulent diffusivity for the mean square fluctuations is approximately equal to the turbulent diffusivity for the mean values of the scalar. The coefficient for turbulent dissipation which was treated as an eigenvalue in the calculation is in good agreement with the expected value.

Nomenclature

a, b = constants of proportionality for rate of growth of mean velocity and species profile respectively, Eqs. (3.2, 3.8)
A_K = Kolmogoroff constant
A_γ = spectral constant for species fluctuations
B = $\langle n'^2_s\rangle(\eta)/\langle n'^2_s\rangle(0)$
c = $(\langle n'^2_s\rangle/\langle n_s\rangle^2)_{r=0}$
d = jet diameter
D_s = diffusion coefficient, species s
ε = dissipation rate of turbulent kinetic energy
H = correction factor
k = wave number
n = total number density
n_s = number density of species s
r, x = radial and axial coordinates
t = time
T = temperature
u, v = velocity in axial and radial directions
$\mathbf{v}$ = mass-average gas velocity
$\mathbf{v}_s$ = average velocity of species s
$\mathbf{V}_s$ = diffusion velocity, species s

Greek symbols

β = constant defined in Eqs. (3.4, 3.7)
ϵ_j = turbulent transport coefficient for j
ζ = $\eta^2 \ln 2$
η = $r/r_{1/2u}$
θ = dimensionless coefficient for turbulent dissipation
λ = microscale
ν = kinematic viscosity
σ = constant defined in Eq. (3.18)
ϕ = dimensionless coefficient for turbulent dissipation
χ = dissipation rate of species fluctuations
ω = $(\phi_\gamma/\phi_s)(b^2/a^2) = \phi_\gamma/\phi_u$

Subscripts and superscripts

$\langle\ \rangle$ = time average
$(\ )'$ = fluctuation
γ = species fluctuations
e = energy containing
0 = initial
s = mean species
t = turbulent
u = mean velocity

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I. Introduction

For turbulent shear flows, there exist few analyses for the fluctuations of a passive scalar such as trace species or temperature. This situation exists primarily because of a lack of understanding of the diffusion of fluctuations and the effect of intermittency on transport. In this paper, a simplified model for the relevant processes is proposed and the results are compared to experiments.

In particular, the use of phenomenological turbulent diffusivities for the generation and diffusion of the fluctuation intensity of a passive species number density is investigated for a round subsonic turbulent jet. These coefficients are treated as parameters in calculations of the self-similar radial profile of the fluctuation intensity. The results are then compared to some experimental profiles to determine the range of values of diffusivities which yield the closest agreement with these experiments. The analysis is performed in terms of a trace number density of a species s; the resulting equations are identical to those for temperature fluctuations.

II. Conservation Equation for the Fluctuation Intensity

In the following, $n_s(\mathbf{r},t)$ is the instantaneous number density of species s; $\langle n_s\rangle(\mathbf{r})$ is the local time-average mean number density, where the time average is long compared to the typical period of turbulent fluctuations, and $n'_s(\mathbf{r}, t)$ is the fluctuation from the mean. The mean square fluctuation $\langle n'^2_s\rangle$ is the time average of $[n'_s(\mathbf{r}, t)]^2$, and its square root is the fluctuation intensity of that species number density, $\langle n'^2_s\rangle^{1/2}$. Since we are interested in these fluctuations, we shall derive the conservation equation for such fluctuations. The instantaneous conservation equation for the s species in the absence of chemical reactions is[1]

$$(\partial n_s/\partial t) + \nabla\cdot(n_s\mathbf{v}_s) = 0 \tag{2.1}$$

where $\mathbf{v}_s(\mathbf{r}, t)$ is the mean velocity of the s species, equal to the sum of the mass-averaged gas velocity $\mathbf{v}(\mathbf{r}, t)$ and the molecular diffusion velocity of this species $\mathbf{V}_s(\mathbf{r}, t)$. This diffusion is caused by instantaneous gradients in species concentration, pressure, and temperature. However, for the subsonic, initially isothermal jet under investigation, only concentration diffusion is important. To simplify the problem, only two species j and s are considered; the ambient gas consists only of j, and the jet is a mixture of s and j. The molecular diffusion is then binary, for which the following expression applies[2]:

$$\mathbf{V}_s = -[n^2 m_j/n_s(n_s m_s + n_j m_j)]D_s\nabla(n_s/n) \quad (2.2)$$

where D_s is the binary diffusion coefficient of s into j, and n is the total number density. The flow is taken to be subsonic and initially isothermal; n and D_s are approximately constant. In addition, either the molecular weights are assumed constant or $n_s \ll n$, which is valid for the experimental data considered herein. Then Eq. (2.1) becomes

$$(\partial n_s/\partial t) + \nabla\cdot(n_s\mathbf{v}) = nD_s\nabla^2(n_s/n) \quad (2.3)$$

Next, the variables in Eq. (2.3) are decomposed into stationary and fluctuating components, $(\ ) = \langle\ \rangle + (\ )'(t)$, so that Eq. (2.3) becomes

$$\frac{\partial n'_s}{\partial t} + \frac{\partial\langle n_s\rangle}{\partial t} + \nabla\cdot(\langle n_s\rangle\langle\mathbf{v}\rangle) + \nabla\cdot(\langle n_s\rangle\mathbf{v}') + \nabla\cdot(n'_s\langle\mathbf{v}\rangle) + \nabla\cdot(n'_s\mathbf{v}') = (\langle n\rangle + n')D_s\nabla^2 \times \left[\frac{\langle n_s\rangle}{\langle n\rangle}\left(1 + \frac{n'_s}{\langle n_s\rangle} - \frac{n'}{\langle n\rangle} - \frac{n'_s n'}{\langle n_s\rangle\langle n\rangle} + \frac{n'^2}{\langle n\rangle^2} + \frac{n'_s}{\langle n_s\rangle}\frac{n'^2}{\langle n^2\rangle}\right)\right] \quad (2.4)$$

with the assumption that $(1 + n'/\langle n\rangle)^{-1}$ can be expanded up to the quadratic term. Now, $n'_s/\langle n_s\rangle \sim 0.2$, while $n'/\langle n\rangle$ is proportional to the square of the Mach number, which is typically 0.1 for the flows being considered; hence, $n'/\langle n\rangle \ll n'_s/\langle n_s\rangle$ and is neglected.

Then upon averaging, one obtains

$$(\partial\langle n_s\rangle/\partial t) + \nabla\cdot(\langle n_s\rangle\langle\mathbf{v}\rangle) + \nabla\cdot\langle n'_s\mathbf{v}'\rangle = \langle n\rangle D_s\nabla^2(\langle n_s\rangle/\langle n\rangle) \quad (2.5)$$

Note that $\langle n'_s\mathbf{v}'\rangle$ is the turbulent transport of the s species. For turbulent jets, it has been found that this diffusion is well represented by a phenomenological diffusivity in those regions where the intermittency is small.[4] We will therefore let

$$\langle n'_s\mathbf{v}'\rangle = \langle n_s\rangle\mathbf{V}_{st} = -\epsilon_s\nabla\langle n_s\rangle \quad (2.6)$$

where ϵ_s is the turbulent coefficient of diffusion for mean species number density, and neglect the last term which represents molecular diffusion. The equation for the fluctuation intensity $\langle n'^2_s\rangle$ is obtained by first subtraction of Eq. (2.5) from Eq. (2.4), then by multiplying by $2n'_s$, and taking the time average as follows for incompressible flows:

$$(\partial\langle n'^2_s\rangle/\partial t) + \nabla\cdot(\langle n'^2_s\langle\mathbf{v}\rangle\rangle) + 2\langle n'_s\mathbf{v}'\rangle\cdot\nabla\langle n_s\rangle + \nabla\cdot\langle n'^2_s\mathbf{v}'\rangle + 2D_s\langle n'_s\nabla^2 n'_s\rangle = 0 \quad (2.7)$$

Equation (2.7) represents the transport equation for the mean square fluctuation of the number density of the s species. The first term is the local temporal change of $\langle n'^2_s\rangle$; the second term is its convection. The third term is the source term [see Eq. (2.6)], which is caused by gradients in the mean value of n_s: the large-scale turbulent velocity field interchanges fluid regions of different local values of n_s which causes fluctuations. The fourth represents turbulent diffusion and the last term represents the destruction of turbulence fluctuations by turbulent dissipation.

Thus, the mean square fluctuation intensity is determined by a balance between generation by large-scale turbulent transport and destruction by diffusive smearing.[5] The diffusion term is conservative and acts to even out the spatial distribution of fluctuations. The last term can be written as

$$D_s\langle\nabla^2 n'^2_s\rangle - 2D_s\langle(\nabla n'_s)^2\rangle$$

As is usual in turbulence theory, the first term of the foregoing which represents the molecular diffusion of fluctuations can be neglected in comparison to turbulent diffusion. The second term represents the destruction of fluctuations by diffusive smearing. It is well accepted that, in shear flows at high Reynolds numbers, this is caused by the smallest eddies which are generally isotropic. One can therefore use the results of isotropic turbulence, and represent this term as[5] (see Appendix)

$$\chi \equiv -2D_s\langle(\nabla n'_s)^2\rangle \approx -(2\langle n'^2_s\rangle/3mA_\gamma)\mathcal{E}^{1/3}k_e^{2/3} \quad (2.8)$$

where A_γ is the Kolmogoroff number for species fluctuations, $\mathcal{E}$ is the rate of destruction of turbulent kinetic energy, and k_e is the energy-containing wave number. The integral scale length for temperature fluctuations is very nearly equal to that for velocity fluctuations, both for isotropic turbulence[6] and turbulent jets.[7] Thus, k_e is taken equal for velocity and species fluctuations, and, from the Appendix,

$$k_e^{2/3} = (3m/\langle q'^2\rangle)A_K\mathcal{E}^{2/3} \quad (2.9)$$

where A_K is the Kolmogoroff constant for the velocity fluctuation spectrum, and $\frac{1}{2}\langle q'^2\rangle$ is the turbulent kinetic energy. Combination of Eq. (2.8) and Eq. (2.9) yields

$$\chi = (2\langle n'_s\rangle\mathcal{E}/\langle q'^2\rangle)(A_K/A_\gamma) \quad (2.10)$$

One may calculate $\mathcal{E}$, the rate of dissipation of turbulent kinetic energy directly from the conservation equation for turbulent kinetic energy using available measurements of the decay of mean velocity and velocity fluctuations. However, the dissipation is due largely to the small-scale eddies which are essentially isotropic, so that we may use expressions for the dissipation derived for isotropic turbulence in terms of the Taylor microscale λ_g. Both methods give approximately the same result for the problem examined herein, but the latter estimate is somewhat simpler, since it gives the ratio of $\mathcal{E}/\langle q'^2\rangle$ directly in terms of the Taylor microscale, so that $\mathcal{E}$ and $\langle q'^2\rangle$ do not have to be estimated separately. In any case the numerical value of the dissipation is an eigenvalue to this problem and does not affect the calculation of the fluctuation intensity of the scalar.

Now, for isotropic turbulence,

$$\epsilon = 5\nu\langle q'^2\rangle/\lambda_g^2 \quad (2.11)$$

where ν is the kinematic viscosity, and λ_g is the Taylor microscale for transverse velocity fluctuations. Thus, Eq. (2.10) becomes

$$\chi = -10\nu\langle n'^2_s\rangle HA_K/\lambda_g^2A_\gamma \quad (2.12)$$

where H is a correction factor for shear flows, since the value of $\mathcal{E}$ given by Eq. (2.11) may require a correction to account for the balance of turbulent kinetic energy.[8,9]

Finally, the turbulent diffusion term in Eq. (2.7) must be considered. The object of the paper is to determine the utility of using a phenomenological turbulent diffusivity, as originally suggested by Boussinesq for the diffusion of turbulence kinetic energy (Ref. 3, p. 297) to the problem of species fluctuation intensity, e.g., let the triple correlation of $\mathbf{v}'$ and n'^2_s be represented by a product of a phenomenological constant coefficient ϵ_γ and the gradient of $\langle n'^2_s\rangle$,

$$\langle\mathbf{v}'n'^2_s\rangle = -\epsilon_\gamma\nabla\langle n'^2_s\rangle \quad (2.13)$$

It is known that, in the transport of heat and kinetic energy in a turbulent wake, the complete transport is not completely represented by such a simple gradient, but should include the bulk transport by large scale motions.[10] Thus, it is of interest to determine whether Eq. (2.13) provides an adequate representation of the transport of fluctuations, as determined by calculations of the profile of $\langle n'^2_s\rangle$ and comparison with experiment. Note that ϵ_s may not be equal to ϵ_γ; another object of the present study is to determine their relationship.†

† From the present analysis, it is not possible to determine the absolute magnitudes of the turbulent diffusivities.

With the use of Eqs. (2.6, 2.12, and 2.13), Eq. (2.7) becomes

$$(\partial/\partial t)\langle n'^2_s\rangle + \nabla\cdot(\langle \mathbf{v}\rangle\langle n'^2_s\rangle) - 2\epsilon_s(\nabla\langle n_s\rangle)^2 - \nabla\cdot\epsilon_\gamma\nabla\langle n'^2_s\rangle = -10\nu\langle n'^2_s\rangle HA_K/\lambda_g{}^2A_\gamma \quad (2.14)$$

In the next section, Eq. (2.14) is specialized to a self-preserving jet and solved for the profile of fluctuation intensities, for various ratios of ϵ_s and ϵ_γ to ϵ_u.

III. Scalar Fluctuation Intensity in a Turbulent Jet

For an incompressible axially symmetric jet, Eq. (2.15) is expressed in cylindrical coordinates. Now, $\partial\langle n_s\rangle/\partial x \ll \partial\langle n_s\rangle/\partial r$ and may be neglected as well as $\partial^2\langle n'^2_s\rangle/\partial x$. Thus, Eq. (2.14) becomes

$$\left[\langle u\rangle\frac{\partial}{\partial x} + \langle v\rangle\frac{\partial}{\partial r} - r^{-1}\frac{\partial}{\partial r}\epsilon_\gamma r\frac{\partial}{\partial r} + \frac{10\nu HA_K}{\lambda_g{}^2A_\gamma}\right]\langle n'^2_s\rangle = 2\epsilon_\gamma\left(\frac{\partial\langle n_s\rangle}{\partial r}\right)^2 \quad (3.1)$$

where $\langle u\rangle(x,r)$ and $\langle r\rangle(x,r)$ are the mean velocities in the axial direction x and radial direction r.

To solve Eq. (3.1), the mean flowfield, mean species distribution, and $\lambda_g{}^2$ must be given. It is well known that the diameter of circular jets grows linearly with axial distance; that is, defining $r_{(1/2)u}(x)$ as the radial coordinate where $\langle u\rangle(r,x) = \frac{1}{2}\langle u\rangle(0,x)$, then

$$r_{(1/2)u}(x) = ax \quad (3.2)$$

where a is a constant. The velocity flowfield is determined by the equations of continuity and momentum. For self-similar jets, the velocity decays as x^{-1} along the axis, and, from measurements of the velocity profile, the phenomenological turbulent kinematic viscosity ϵ_u is approximately constant.[4] For ϵ_u constant, the velocity profile is given by[4]

$$\langle u\rangle(r,x)/\langle u\rangle(0,x) = [1 + K(r/r_{(1/2)u})^2]^{-2} \quad (3.3)$$

but for the present purposes, the axial velocity profile can be taken as a Gaussian, since there is very little difference between the two,

$$\langle u\rangle(r,x) = (U_0\beta_u d/x)\exp[-(\ln 2)(r/r_{(1/2)u})^2] \quad (3.4)$$

where U_0 is the initial jet speed. The relation between β_u, a, and ϵ_u is most easily determined by satisfying the momentum equation on the axis,

$$(\langle u\rangle\partial\langle u\rangle/\partial x)_{r=0} = [r^{-1}(\partial/\partial r)(r\epsilon_u\partial\langle u\rangle/\partial r)]_{r=0} \quad (3.5)$$

In terms of a nondimensional turbulent kinematic viscosity, $\phi_u = \epsilon_u/U_0d$, substitution of Eqs. (3.2) and (3.4) in Eq. (3.5) yields

$$\beta_u = (4\phi_u \ln 2)/a^2 \quad (3.6)$$

In the same way, we let the mean species number density be given by

$$\langle n_s\rangle = (n_0\beta_s d/x)\exp[-(\ln 2)(r/r_{(1/2)s})^2] \quad (3.7)$$

where

$$r_{(1/2)s} = bx \quad (3.8)$$

and n_0 is the initial number density of s in the jet. The relation between the nondimensional eddy diffusivity, $\phi_s = \epsilon_s/U_0d$, β_s, and b is obtained by satisfying the mean-s species conservation equation (2.5) on the axis, neglecting molecular diffusion,

$$(\langle u\rangle\partial\langle n_s\rangle/\partial x)_{r=0} = [r^{-1}(\partial/\partial r)(r\epsilon_s\partial\langle n_s\rangle/\partial r)]_{r=0} \quad (3.9)$$

from which

$$\beta_u = (4\phi_s \ln 2)/b^2 \quad (3.10)$$

giving the familiar result that the ratio of the spreads b^2/a^2 is equal to ϕ_s/ϕ_u. Although ϕ_u is approximately constant, ϕ_s decreases in the radial direction.[4] In the rest of the analysis, ϕ_u, ϕ_s, and $\phi_\gamma = \epsilon_\gamma/U_0d$ are all treated as constant, since this will serve to illustrate the primary effect with very little loss generality.

Finally, it is necessary to express λ_g in terms of the turbulent jet parameters. For self-preserving isotropic turbulence, $\lambda_g{}^2 \approx 10\,\nu t$. In a circular jet, the velocity fluctuations decrease inversely with the downstream distance while the microscale increases linearly with downstream distance, so that the turbulence Reynolds number remains approximately constant. Thus the flow is essentially self-preserving; that is, the ratio of various turbulence scale sizes remains constant. Under these conditions, one would expect this relation for self-preserving isotropic turbulence to be applicable also to the jet. Time has been arbitrarily determined on the axis yielding a value that agrees with measured values of λ_g in a jet[11] to within 20%, which is sufficiently accurate for the present purposes. Thus,

$$\lambda_g{}^2 \cong 10\nu\int_0^x [\langle u\rangle(0,x)]^{-1}dx \quad (3.11)$$

Use of Eq. (3.3) and integration yields

$$\lambda_g \cong (5\nu/\beta_u U_0 d)^{1/2}x \quad (3.12)$$

Equation (3.1) permits a self-similar solution; that is, $\langle n'^2_s\rangle = X(x)B(\eta)$ where $\eta = r/r_{(1/2)u}$, when $X(x) \sim x^{-2}$. Since $\langle n_s\rangle \sim x^{-1}$, for self-similarity, the ratio of the fluctuation intensity $\langle n'^2_s\rangle^{1/2}$ to the mean $\bar{n}_s$ is constant on the axis. After several x/d, such a self-similar profile is achieved.[7,12] We therefore let

$$\langle n'^2_s\rangle = c^{-2}n_0{}^2\beta_s{}^2d^2x^{-2}B(\eta) \quad (3.13)$$

where $B(\eta)$ is the radial profile of mean square fluctuation intensities, and c is the ratio of fluctuation intensity to mean species density on the axis

$$c = [\langle n'^2_s\rangle^{1/2}/\langle n_s\rangle]_{r=0} \quad (3.14)$$

Substitution of Eqs. (3.4, 3.6–3.8, 3.10, 3.12, and 3.13) into Eq. (3.1) yields the following ordinary differential equation for $B(\zeta)$:

$$\omega B'' + (\omega - 1 + e^{-\zeta})B' + (2e^{-\zeta} - \theta)B + \sigma\exp[-2a^2b^{-2}\zeta] = 0 \quad (3.15)$$

where

$$\zeta = \eta^2 \ln 2 \quad (3.16)$$

$$\omega = \phi_\gamma b^2/\phi_s a^2 = \phi_\gamma/\phi_u \quad (3.17)$$

$$\sigma = 2a^2/b^2c^2 \quad (3.18)$$

$$\Theta = 2HA_K/A_\gamma \quad (3.19)$$

The boundary conditions are

$$B(0) = 1 \qquad \lim_{\zeta\to\infty} B(\zeta) = 0$$

If ω, σ, and Θ were known, the solution of Eq. (3.15) would yield the profile of fluctuation intensities. Since the present purpose is to determine the value of ϕ_γ which yields the closest agreement with the experimental profile, ω, σ, and Θ were treated as parameters. It can easily be shown that specification of these three parameters and the two boundary conditions overspecifies the problem; therefore either one of the parameters must be left as an eigenvalue to the problem or one of the boundary conditions must be left free. For convenience, Θ was treated as the eigenvalue, and the second boundary condition was replaced by the value of B near the turbulent front since it is not expected that Eq. (3.15) applies in regions of large intermittency.

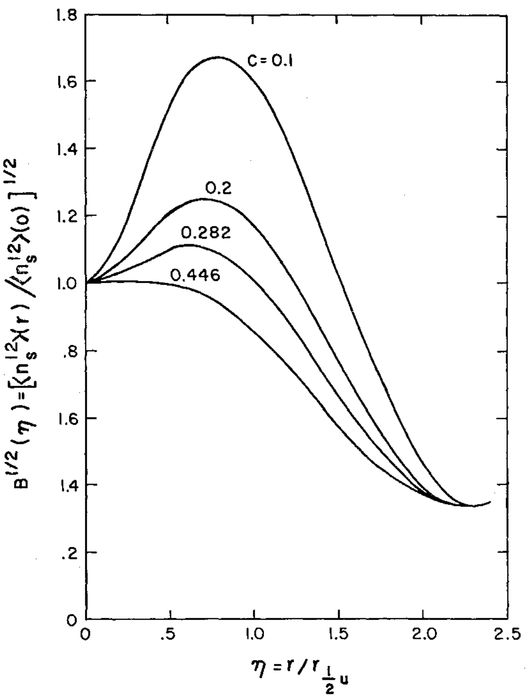

Fig. 1 Turbulent intensity profiles $B^{1/2}(\eta)$ for $a = b$, $\omega = 1$.

Equation (3.15) was solved iteratively on an electronic digital computer for the following values of the parameters: $(b/a)^2 = 1, 1.5, 2$; $\omega = 0.5, 1, 2, 3$; $\sigma = 10, 25, 50, 200$. The results of some typical calculations are shown in Fig. 1. It can be seen that for small values of σ, corresponding to large fluctuations relative to the source of fluctuations, the profile is close to gaussian. However, as σ is increased, e.g., increasing the source strength relative to the level of fluctuations, a maximum is caused to appear at the point where the source strength $(\partial\langle n_s\rangle/\partial r)$ is a maximum. The relative height of this maximum is shown in Fig. 2. The experimentally determined relative maximum is about 1.25; thus, for this value, ω is quite sensitive to c.

Typical eigenvalues Θ are shown in Fig. 3. It can be seen that Θ is approximately inversely proportional to c; that is, the higher the dissipation rate, the lower the value of the turbulent fluctuation intensity, which is the expected result. It can also be seen that the value of Θ is insensitive to the value of ω. This is also as expected, since the diffusion term is conservative, and does not grossly affect the relation between dissipation and the level of fluctuations.

The profiles shown in Fig. 1 which are for $a = b$, e.g., $\phi_s = \phi_u$, exhibit a tail for $\eta > 2$, which is a consequence of both

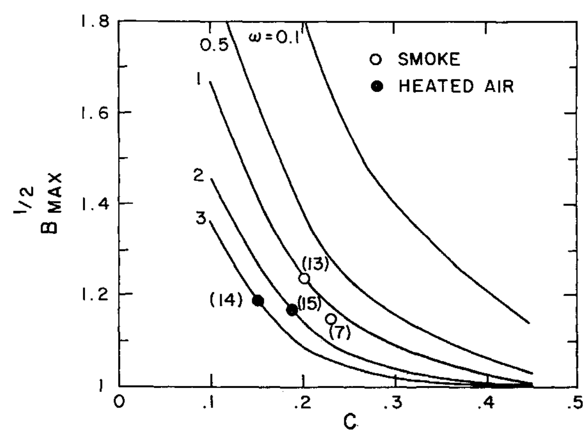

Fig. 2 Variation of the maximum fluctuation intensity with ratio of fluctuation intensity to mean, and turbulent diffusivity for fluctuations, for $a = b$.

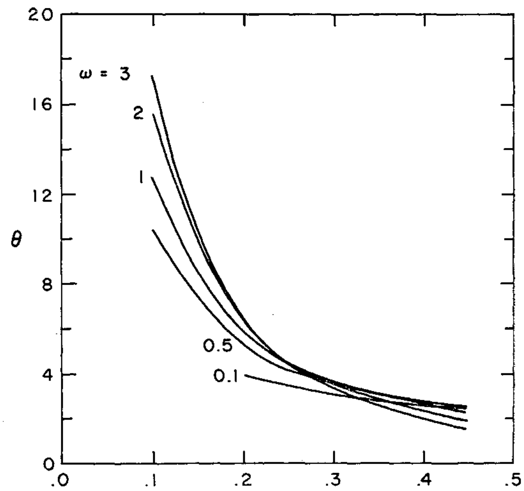

Fig. 3 Variation of the dimensionless turbulent dissipation rate for $a = b$.

having a fixed outer boundary condition and having the relative maximum of $\partial\bar{n}_s/\partial r$ close to the axis relative to the velocity profile. This tail disappears as b/a is increased; in fact, for $b/a = 2$, the slope $\partial B/\partial\eta$ is larger than the experimental values for $\eta > 2$. Figure 4 shows the variation in profile with b/a for $\phi_\gamma = \phi_s$ and $c = 0.2$. For these conditions, the height of the maximum varies very little with (b/a). Also, the value of Θ is insensitive to (b/a), being 5.845 for $(b/a)^2 = 1$ and 5.617 for $(b/a)^2 = 2$.

IV. Comparison with Experiment

To determine the applicable range of ω, the ratio of turbulent diffusion coefficient for fluctuations to eddy diffusivity, the available experimental measurements were examined. These experiments were conducted in a jet which contained smoke particles which acted as a tracer for the jet effluent[7,12] and a heated jet.[13,14] The experimentally determined values of the fluctuation ratio on the axis and the maximum fluctuation intensity are shown on Fig. 2. The two smoke jet measurements give about the same value of ω, that is, about unity. For this value, the computed fluctuation intensity radial profile fits the experimental profile of Ref. 12 very well as shown in Fig. 4 for $(b/a)^2 = 1.5$. From the available experimental data for mean velocity in a jet[15] and for scalar

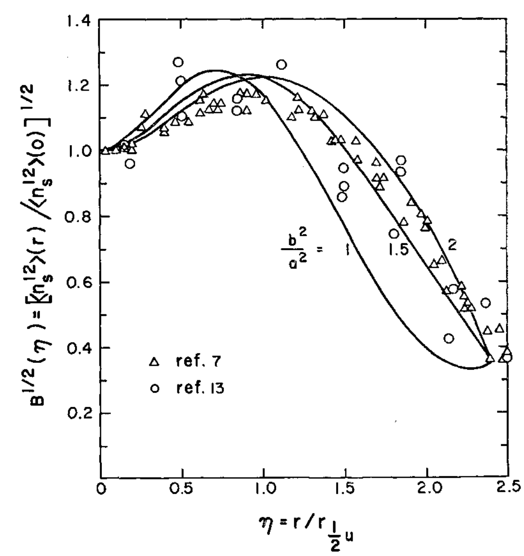

Fig. 4 Fluctuation intensity profiles for $\phi_s = \phi_\gamma$, $c = 0.2$ and various values of b/a compared with smoke jet data; triangles, Ref. 7; circles, Ref. 13.

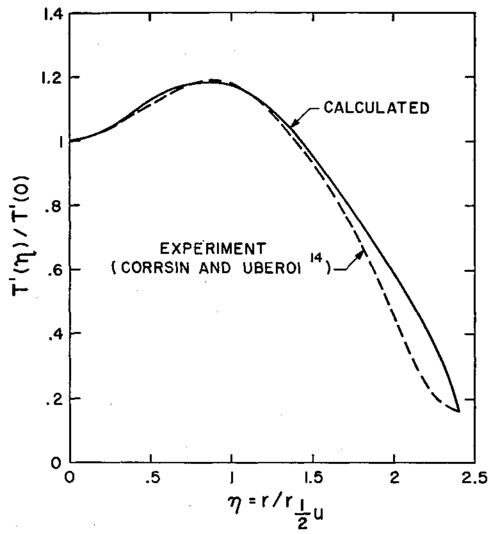

Fig. 5 Temperature fluctuations in a heated (170°C) jet at $x/d = 20$, compared with calculation for $b^2/a^2 = 1.5$, $\phi_s/\phi_\gamma = 3$, $c = 0.13$.

additives,[7] b is about 25% larger than a; thus, the theoretically predicted profile is in very close agreement with experimental. Increasing the value of ω and c slightly would improve the fit of the theoretical curve in Fig. 4 with the more recent data[7] for a smoke jet.

It is also of interest to compare the predicted eigenvalue of Θ of about 5.7 for $(B_{\max})^{1/2} = 1.25$, $\phi_s = \phi_\gamma$, and $c \cong 0.2$ to the expected value. From measurements of the balance of turbulent kinetic energy in a jet,[8] $H \approx 2$, whereas Gibson[9] has esimated that $A_K = 1.6$. From the spectral measurements of Ref. 4, it appears that $A_\gamma \approx 1.1$. Thus, the expected value of Θ is about 5.8, which matches the theoretical eigenvalue probably better than should be expected.

It should be noted that the comparison with smoke data assumed an appreciable inertial convective subregion for both the velocity and species fluctuation spectra. For a lower Reynolds number and large molecular Schmidt number, there can be an appreciable range of viscous convection,[16] for which $E_\gamma(k) \sim k^{-1}$, and Eq. (2.9) may not be applicable.

In principle, the results of this analysis should also be applicable to temperature fluctuations in a heated jet, provided that the temperature differences are small. Measurements of temperature fluctuations in a heated jet (initial temperature 170°C[13]) indicate that the relative maximum of temperature fluctuations is about 1.15, e.g., about the same as for smoke, but the ratio of temperature fluctuations to mean on the axis is only about 0.125 as compared to 0.18 for smoke, at an x/d of 20. Figure 2 indicates that for $c \approx 0.125$, $\omega = 3$, that is, $\phi_\gamma/\phi_s = 3$. Using these two values and $b^2 = 1.5a^2$, corresponding to the measured mean profiles,[13,15] the calculated temperature fluctuation profile is shown in comparison to the experimental data[13] in Fig. 5. The eigenvalue Θ is equal to 11.111, about double that for the smoke jet. There is some question regarding the calibration of the instrumentation used, but, in addition, it appears that for a heated jet the asymptotic level of fluctuations is not reached until $x/d \gtrsim 50$.[14] Thus, the comparison of the data on Ref. 13 with the present theory is probably not valid. More recent data[14] in a heated jet (initial temperature 225°) give closer agreement with the recent smoke jet data,[7] in that the asymptotic ratio of fluctuation to mean temperature on the axis is 0.18, with a maximum value of $B^{1/2}$ of about 1.17. From Fig. 2, this yields a value of ω of about 2.

All of the preceding data may be represented by $\omega \approx 1.5 \pm 0.5$, whereas from Ref. 7 the average spread of the mean scalar to the velocity profile is given by $b^2/a^2 \equiv \phi_s/\phi_u$ of about 1.5. From Eq. (3.17) one may conclude that $\phi_\gamma \approx \phi_s$, that is, the phenomenological turbulent diffusion coefficient for fluctuations of the scalar is approximately equal to that of the scalar mean. Conversely, with the use of these parameters, it is possible to calculate the scalar fluctuation intensity.

V. Conclusions

The diffusion of the mean square turbulent fluctuations of a scalar was represented by gradient diffusion, using a turbulent diffusivity. By comparison of the calculated profiles for the scalar fluctuation intensity with measured profiles in low Mach number turbulent jets it appears that this diffusivity is equal to the turbulent diffusivity for the mean scalar within 30%. Furthermore, the calculated turbulent dissipation, which was treated as an eigenvalue, gives good agreement with the expected value. For the types of turbulent jets considered, it therefore appears to be possible to calculate the turbulent fluctuation intensity of a scalar by the use of these parameters.

Appendix: Turbulent Dissipation of a Scalar

The turbulent dissipation rate for a passive scalar is given by[17]

$$\chi = -2D_s\langle(\nabla n'_s)^2\rangle = 12D_s\langle n'^2_s\rangle/\lambda_\gamma^2 \tag{A1}$$

The spectrum E_γ in the inertial subrange is given by[17]

$$E_\gamma(k) \approx A_\gamma \chi_\gamma \mathcal{E}^{-1/3} k^{-5/3} \tag{A2}$$

where A_γ is a constant of order unity. However,

$$\langle\gamma'^2\rangle = \int_0^\infty E(k)dk \approx m \int_{k_e}^\infty E_\gamma(k)dk \tag{A3}$$

where m is a factor to account for the portion of the spectrum for which $0 < k < k_e$. Substitution of Eq. (A2) into Eq. (A3), integration, and rearrangement yields

$$\chi = (2/3m)(\langle\gamma'^2\rangle/A_\gamma)\mathcal{E}^{1/3}k_e^{2/3} \tag{A4}$$

If $mA_\gamma \approx \frac{2}{3}$, Eq. (A4) corresponds to the expressions given by Corrsin[19] for a Schmidt number of unity. We may likewise assume a Kolmogoroff spectrum for velocity fluctuations,

$$E = A_K \mathcal{E}^{2/3} k^{-5/3} \tag{A5}$$

where A_K is the Kolmogoroff constant. Hence

$$\tfrac{1}{2}\langle q'^2\rangle = \int_0 E dk = \frac{3}{2} m A_K \mathcal{E}^{2/3} k_e^{-2/3} \tag{A6}$$

Since the energy-containing wave number for velocity fluctuations is approximately the same as for the species fluctuations[7] then $m \approx n$, and substitution of Eq. (A6) into Eq. (A5) yields

$$\chi = (2\langle\gamma'^2\rangle\mathcal{E}/\langle q'^2\rangle)(A_K/A_\gamma) \tag{A7}$$

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Aerodynamics of Bodies of Revolution in Coning Motion

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Nonlinear functional analysis is used to show that the moments for an arbitrary motion about the center of gravity may be compounded of the sum of contributions from 4 simple motions, 3 of which are well known. The fourth, coning motion, is investigated experimentally with an apparatus designed to reproduce the coning motion in a wind-tunnel. Photographs of the vortices on the leeward side of the body reveal that they become asymmetrically disposed with respect to the angle-of-attack plane during coning motion. This makes them a potent source of side moment. The contribution of the vortices to side moment appears to be proportional to their contribution to pitching moment during steady planar motion multiplied by the turning rate of the angle-of-attack plane during coning motion. The moment is a potential causative agent in the occurrence of large-amplitude circular limit motions.

Nomenclature

C_l = rolling-moment coefficient, L/q_0Sl

C_m = pitching-moment coefficient, M/q_0Sl

C_{m_v} = contribution to pitching-moment coefficient attributable to presence of vortices

$C_{m_\sigma}[\sigma(\xi),\varphi(\xi),\dot\psi(\xi);t,\tau]$ = indicial pitching-moment response measured at t/unit step change in σ occurring at τ with φ and $\dot\psi$ held fixed at $\varphi(\tau)$, $\dot\psi(\tau)$

$C_{m_{\dot\varphi}}[\sigma(\xi),\varphi(\xi),\dot\psi(\xi);t,\tau]$ = indicial pitching-moment response measured at at t/unit step change in $\dot\varphi l/V_0$ occurring at τ with σ and $\dot\psi$ held fixed at $\sigma(\tau)$, $\dot\psi(\tau)$

$C_{m_{\dot\psi}}[\sigma(\xi),\varphi(\xi),\dot\psi(\xi);t,\tau]$ = indicial pitching-moment response measured at t/unit step change in $\dot\psi l/V_0$ occurring at τ with σ and φ held fixed at $\sigma(\tau)$, $\varphi(\tau)$

C_n = side-moment coefficient, N/q_0Sl

$G[u(\xi),\nu(\xi),w(\xi)]$ = functional notation: value at $\xi = t$ of a function $F(t)$ which depends on all the values taken by the three argument functions $u(\xi)$, $v(\xi)$, $w(\xi)$ over the interval $0 \leq \xi \leq t$

l = characteristic length

l_0 = characteristic length in experiment, distance along axis of symmetry from center of gravity to nose of body, Fig. 3

L = moment about axis of cylindrical symmetry, Fig. 1

M = moment about an axis normal to the plane of resultant angle of attack and along line of nodes, Fig. 1

M_0 = Mach number

N = moment about an axis in the plane of resultant angle of attack and normal to line of nodes, Fig. 1

p, q, r = components of angular velocity about x,y,z axes, respectively

q_0 = dynamic pressure, $\frac{1}{2}\rho_0 V_0^2$

Re = freestream unit Reynolds number, $\rho_0 V_0/\mu$

S = characteristic area (body base area in experiment)

t = time

V_0 = flight speed

X_7, Y_7, Z_7 = axes having fixed directions in space, origin at center of gravity, Fig. 1

x, y, z = axes fixed in body, origin at center of gravity, Fig. 1

γ = angle between plane of resultant angle of attack and plane of symmetry of vortices,

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